Orders and Chow groups

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Arakelov Geometry is a combination of Algebraic geometry (schemes) and Hermitian complex geometry. Its main achievement today is the proof of the Mordell conjecture. In this paper we will introduce some tools used in Arakelov Geometry over reduced orders. We will discuss in the first section properties of reduced orders, and introduce Arithmetic Chow groups of such objects (a slight refinement of the classical Chow groups in Algebraic Geometry).

1 Orders

Through all this paper, all rings are supposed commutative with unit. In this section, we will introduce a class of rings called Orders.

1.1 Preliminaries

Lemma 1.1. *Let A, B be rings, $f : A \mapsto B$ a ring homomorphism, $b \in B$. The following are equivalent:*

(i) $\exists a_0, \ldots, a_n \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$$

(ii) the A-sub algebra $A[b]$ generated by b is a finitely generated A-module.

(iii) There exists a finitely generated A-submodule $M = A\omega_1 + \cdots + A\omega_n$ of B such that $1 \in M$ and $bM \subset M$.

Proof. We prove the following implications:

(i) ⇒ (ii): Let $g \in A[X]$ be a monic polynomial such that

$$g(b) = b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0 \tag{1}$$

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Let M be the finitely generated A-submodule $M = A + Ab + \cdots + Ab^{n-1} \subset A[b]$. By 1 we have that $b^n \in M$, multiplying by b gives $b^{n+1} \in M$ hence inductively, $b^k \in M\ \forall k \geq 1$. Thus $M = A[b]$ is finitely generated.

(ii) $\Rightarrow$ (iii): Choose $M = A[b]$.

(iii) $\Rightarrow$ (i): Let M be as in *(iii)*. Let $(a_{i,j}) \in M_n(A)$ such that

$$b\omega_i = \sum_{i=1}^{n} a_{ij}\omega_j$$

Let $C = (b\delta_{ij} - a_{ij})$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$

One has

$$C \cdot \omega = \begin{bmatrix} \sum_{j=1}^n b\delta_{1j}\omega_1 - \sum_{j=1}^n a_{1j}\omega_j \\ \vdots \\ \sum_{j=1}^n b\delta_{nj}\omega_1 - \sum_{j=1}^n a_{nj}\omega_j \end{bmatrix} = \begin{bmatrix} a\omega_1 - a\omega_1 \\ \vdots \\ a\omega_n - a\omega_n \end{bmatrix} = 0$$

Crammer's rule gives

$$\det(C) \cdot \omega = C^{ad} C\omega = 0 \ \Rightarrow\ \det(C) \cdot \omega_k = 0\ \forall k \geq 1$$

Hence $\det(C) \cdot M = 0$ and $\det(C) = 0$ since $1 \in M$. Finally, consider

$$\begin{aligned} g(X) &= \det(\delta_{ij}X - a_{ij}) \in A[X] \\ &= X^n + a_{n-1}X^{n-1} + \cdots + a_0 \end{aligned}$$

Then $g(b) = \det(C) = 0$.

□

Corollary 1.2. *Let R be a ring, suppose that R is a finitely generated free $\mathbb{Z}$-module, of rank n and $S = \{x \in R,\ x \text{ is regular}\}$. Then R is integral over $\mathbb{Z}$.*
Moreover if $K := S^{-1}R$ denotes the total quotient ring of R then

$$K = R \otimes_{\mathbb{Z}} \mathbb{Q}$$

Proof. By Lemma (1.1) for $a \in R$, $\exists c_0, \ldots, c_n \in \mathbb{Z}$ such that

$$a^n + c_{n-1}a^{n-1} + \cdots + c_0 = 0$$

If $a \in S$ then $\varphi_a : x \mapsto ax$ is injective (Otherwise $\exists u, v \in R, u \neq v$ and $au = av \Rightarrow a(u - v) = 0$ which is impossible.) hence $c_0 \neq 0$ and for

$$b = a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_1$$

One gets

$$ab = -c_0 \in \mathbb{Z} \setminus \{0\}$$

Hence

$$S = \mathbb{Z} \setminus \{0\} \text{ and } R \hookrightarrow S^{-1}R \cong R \otimes_{\mathbb{Z}} (\mathbb{Z} \setminus \{0\})^{-1}\mathbb{Z} \cong R \otimes_{\mathbb{Z}} \mathbb{Q}$$

□

Definition 1.3. *An order is a ring $\mathcal{O}$ that is also a finitely generated $\mathbb{Z}$-module. It is called an integral order if $\mathcal{O}$ is an integral domain.*

Example 1.4.

- *a lattice $\Lambda = \{\sum a_i x_i,\ a_i \in \mathbb{Z}\}$ is an order.*
- *$\mathbb{Z}[i] = \{a + ib,\ a, b \in \mathbb{Z}\}$ is an integral order.*
- *In general, for a number field K, the ring of integers*

$$\mathcal{O}_K = \{x \in K \mid x \text{ integral over } \mathbb{Z}\} \tag{2}$$

is an integral order (by Lemma 1.1), called the maximal order in K.

Here is an important property about special cases of orders.

Proposition 1.5. *Let $\mathcal{O}$ be an order.*

(i) $\mathcal{O}$ is reduced $\Leftrightarrow$ there exists algebraic number fields $K_1, \dots, K_n$ and a ring homomorphism $\varphi : \mathcal{O} \hookrightarrow \bigoplus_{i=1}^n \mathcal{O}_{K_i}$

(ii) $\mathcal{O}$ is normal $\Leftrightarrow$ there exists algebraic number fields $K_1, \dots, K_n$ and a ring isomorphism $\varphi : \mathcal{O} \xrightarrow{\sim} \bigoplus_{i=1}^n \mathcal{O}_{K_i}$

Before proving this proposition, we need the following lemma:

Lemma 1.6. *Let $\mathcal{O}$ be a reduced order, $K = S^{-1}\mathcal{O}$ its total quotient ring.*

(i) K is an Artinian ring, and one has

$$K \cong \bigoplus_{i=1}^{n} K_{\mathfrak{m}_i}$$

Where $\mathfrak{m}_i \in Spec(K)$. Moreover, $K_{\mathfrak{m}_i} = \mathcal{O}_{\mathfrak{q}_i}$ for $\mathfrak{q}_i = \mathcal{O} \cap \mathfrak{m}_i$.

(ii) Let $K(\mathbb{C}) = Hom_R(K, \mathbb{C})$ and $K_i(\mathbb{C}) = Hom_R(K_i, \mathbb{C})$. For $\tau \in K_i(\mathbb{C})$ let $\overline{\tau} \in K(\mathbb{C})$ be given by

$$\overline{\tau}(x_1, \dots, x_n) = \tau(x_i)$$

. Then one has

$$\coprod_{i=1}^{n} K_i(\mathbb{C}) \cong K(\mathbb{C})$$

. In particular, $\#K(\mathbb{C}) = dim_{\mathbb{Q}}(K_i)$.

Proof. (i) By exactness of localization, K is Noetherian. Let $\mathfrak{p}_1, \dots \mathfrak{p}_n$ be minimal primes of K, since K is also reduced (exactness of localization again)

$$Nil(K) = \bigcap_{i=1}^{n} \mathfrak{p}_i = (0)$$

Let $\mathfrak{p} \in Spec(K)$, be a proper prime ideal, and $x \in \mathfrak{p}$. Then x is a zero divisor, hence $\exists y \neq 0$ such that $xy = 0$. In particular, $xy \in \mathfrak{p}_i$ for all $i = 1, \ldots, n$.
If $x \notin \mathfrak{p}_i \forall i$ then $y \in \bigcap \mathfrak{p}_i = (0)$ which is impossible. Hence $\exists i$ such that $x \in \mathfrak{p}_i$ and

$$\mathfrak{p} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$$

By the prime avoidance lemma, $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some i, thus $\mathfrak{p} = \mathfrak{p}_i$. Hence all the $\mathfrak{p}_i$ are maximal and as K is Noetherian, we conclude that it is Artinian.
Now, recall from Lemma 1.6 in [1] that one has the decomposition

$$K \cong \bigoplus_{i=1}^{n} K_{\mathfrak{m}_i}$$

As $\mathcal{O}$ is integral and flat over $\mathbb{Z}$, one has that $\forall \mathfrak{q}$ minimal prime in R, $\mathfrak{q} \cap \mathbb{Z} = \{0\}$ (since $\mathbb{Z}$ is an integral domain) and so $\mathfrak{q} \cap S = \emptyset$, thus $\mathfrak{q}_i = \mathcal{O} \cap \mathfrak{m}_i$ are minimal primes in $\mathcal{O}$. Finally,

$$Nil(K_{\mathfrak{m}_i}) = \bigcap_{p \in mSpec(K_{\mathfrak{m}_i})} \mathfrak{p} = \mathfrak{m}_i K_{\mathfrak{m}_i} = Nil(K)_{\mathfrak{m}_i} = (0)K_{\mathfrak{m}_i} = (0)$$

Hence $K_{\mathfrak{m}_i}$ are fields.

(ii) Let $\tau \in K_i(\mathbb{C})$ such that $\overline{\tau}(x_1, \ldots, x_n) = \tau(x_i)$ then $\overline{\tau} \restriction_{K_i} = \tau$ and $\overline{\tau}(K_j) = 0$ for $i \neq j$. Thus the map

$$\tau \longmapsto \overline{\tau} = \begin{cases} \overline{\tau} \restriction_{K_j} = 0 & \text{if } i \neq j \\ \overline{\tau} \restriction_{K_i} = \tau & \text{if } i = j \end{cases}$$

is injective. Now let $e_i = (0, \ldots, 1, \ldots, 0), \sigma \in K(\mathbb{C})$. Since $e_i^2 = e_i$, $\sigma(e_i^2) = \sigma(e_i) \Rightarrow \sigma(e_i) = 1$ (since $\sigma(e_i)$ can not be 0 as $\sigma(1) = 1$. Thus $\exists i$ such that $\sigma(e_i) = 1$, it is unique since for $\sigma(e_j) = 1$ with $i \neq j$ one has

$$0 = \sigma(e_i e_j) = \sigma(e_i)\sigma(e_j) = 1$$

Now let $\tau = \sigma \restriction_{K_i}$, then $\overline{\tau} = \sigma$.

□

Now we prove Proposition 1.5:

Proof proposition 1.5. (i) $\Rightarrow$) Suppose $\mathcal{O}$ is a reduced order.
For simplicity, we note $K_{\mathfrak{m}_i}$ simply by K_i. Consider the following diagram :

$$\begin{array}{ccc} \mathcal{O} & \overset{\exists \phi_i}{\dashrightarrow} & \mathcal{O}_{K_i} \\ \big\downarrow & & \big\downarrow \\ K \cong \bigoplus_{i=1}^{n} K_i & \longrightarrow & K_i \end{array}$$

Then one has an induced map $\exists\varphi : \mathcal{O} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{K_i}$.
Suppose

$$\begin{aligned} \varphi(x) = 0 \ \text{ in } \ \bigoplus_{i=1}^{n} \mathcal{O}_{K_i} &\Rightarrow \exists i \ \text{ such that } \ \phi_i(x_i) = 0 \ \text{ in } \ \mathcal{O}_{K_i} \\ &\Rightarrow x_i = 0 \ \text{ in } \ K_i \ \ \forall i \\ &\Rightarrow x = 0 \ \text{ in } \ \bigoplus_{i=1}^{n} K_i \cong K = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

As $\mathcal{O}$ is flat over $\mathbb{Z}$,

$$0 \longrightarrow \mathcal{O} \hookrightarrow \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = K$$

Thus $x = 0$ in $\mathcal{O}$ and φ is injective.

$\Leftarrow$) Suppose $\exists\varphi : \mathcal{O} \hookrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{K_i}$ and consider the following diagram

$$\begin{array}{ccc} \mathcal{O} & \hookrightarrow & \bigoplus_{i=1}^{n} \mathcal{O}_{K_i} \\ \downarrow & & \downarrow \\ \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = K & \xrightarrow{\sim} & \bigoplus_{i=1}^{n} K_i \end{array}$$

Let $x \in \mathcal{O}$ such that $x^m = 0$ for some $m \geq 1$, then $x_i^m = 0$ in K_i for all $i = 1, \ldots, n$. Since K_i are number fields,

$$x_i^m = 0 \Leftrightarrow x_i = 0$$

Thus $\phi_i(x_i) = 0$ for all $i = 1, \ldots, n$ and $\varphi(x) = 0$ in $\bigoplus_{i=1}^{n} \mathcal{O}_{K_i}$. Since φ is injective by assumption, x must be 0 hence $\mathcal{O}$ has no non-zero nilpotent elements.

(ii) $\Rightarrow$) Suppose $\mathcal{O} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{K_i}$ and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be minimal prime ideals of $\mathcal{O}$, then if we localise at any $\mathfrak{q}_i$, $\mathcal{O}_{\mathfrak{q}_i} = \mathcal{O}_{K_i}$ is integrally closed by definition of $\mathcal{O}_{K_i}$, hence $\mathcal{O}$ is normal.

$\Leftarrow$) Now suppose $\mathcal{O}$ is normal, and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be minimal prime ideals of $\mathcal{O}$. Recall that $\mathcal{O}_{\mathfrak{q}_i} = K_i$ and $K \cong \bigoplus_{i=1}^{n} K_i$. By (i) (normal rings are reduced) we have an injection

$$\varphi : \mathcal{O} \hookrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{K_i}$$

To show our isomorphism, it suffices to show that for all $\mathfrak{p} \in Spec(\mathcal{O})$

$$\varphi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} \xrightarrow{\sim} (\bigoplus_{i=1}^{n} \mathcal{O}_{K_i})_{\mathfrak{p}}$$

Let $\mathfrak{p} \in Spec(\mathcal{O})$, as $\mathcal{O}_{\mathfrak{p}}$ is an integral domain, $\exists! \mathfrak{q}_i \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal. Thus for all $j \neq i$

$$\mathcal{O} \setminus \mathfrak{p} \cap \mathfrak{q}_j \neq \emptyset \ \text{ and } \ \forall x \in \mathfrak{q}_j, x = 0 \ \text{ in } \ \mathcal{O}_{\mathfrak{q}_j}$$

Hence

$$(K_j)_{\mathfrak{p}} = \{0\}$$

And

$$\varphi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} \longrightarrow (\mathcal{O}_{K_i})_{\mathfrak{p}}$$

which is clearly in isomorphism since $R_{\mathfrak{p}}$ is an integral domain.

□

We need a little lemma in order to prove our final and main result of this section.

Lemma 1.7. *Let M be a finitely generated free $\mathbb{Z}$-module and let ϕ be an injective homomorphism. Then*

$$\#\left({}^{M}/_{\phi(M)}\right) = |\det(\phi)|$$

Proof. Let $\{\omega_1, \dots, \omega_n\}$ a $\mathbb{Z}$-basis, $\phi \in End(M)$ such that

$$\phi(w_i) = \sum_{j=1}^{n} b_{ij} w_j \quad , \quad B = (b_{ij}) \in M_n(\mathbb{Z})$$

By the Smith normal form,

$$\exists P, Q \in Gl_n(\mathbb{Z}) \ \text{ such that } \ PBQ = diag(c_i) := C \in M_n(\mathbb{Z})$$

Now let f_P, f_Q, f_C the respective endomorphisms of M given by P, Q and C, then we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow[f_Q]{\sim} & M \\ {\scriptstyle f_C}\downarrow & & \downarrow{\scriptstyle \phi} \\ M & \xleftarrow[f_P]{\sim} & M \end{array}$$

Thus

$$\#\left({}^{M}/_{\phi(M)}\right) = \#\left({}^{M}/_{f_C(M)} \cong \bigoplus_{i=1}^{n} {}^{\mathbb{Z}}/_{(c_i)}\right) = |c_1 \dots c_n|$$
$$= |\det(f_C)| = |\det(f_P)\det(f_C)\det(f_Q)| = |\det(\phi)|$$

Since $\phi(\omega) = B \cdot \omega$. □

Theorem 1.8 (product formula)**.** *Let $\mathcal{O}$ be a reduced order, $K = S^{-1}\mathcal{O}$ be its total quotient ring. then for $x \in k^{\times}$*

$$\prod_{\mathfrak{p} \in mSpec(\mathcal{O})} \#\left({}^{\mathcal{O}}/_{\mathfrak{p}}\right)^{-ord_{\mathfrak{p}}(x)} \prod_{\sigma \in K(\mathbb{C})} |\sigma(x)| = 1$$

Proof. Suppose x is regular ($x \in S$) then $\phi_x : a \mapsto ax$ is an injective ring homomorphism, and by Lemma (1.7)

$$\#\left({}^{\mathcal{O}}/_{x\mathcal{O}}\right) = |\det(\phi_x)|$$

Let K_i be number fields such that

$$K \cong \bigoplus_{i=1}^{n} K_i$$

Let $x = (x_1, \dots, x_n)$ and

$$\phi_x(a) = (x_1 a_1, \dots, x_n a_n)$$

Let $\phi_i \in End(K_i)$ given by $\phi_i(a_i) = x_i a_i$, then

$$\det(\phi_x) = \prod_{i=1}^{n} \det \phi_i$$

Recall that since K_i are algebraic number fields, one has that

$$N_{K_i(\mathbb{C})}(x_i) = \det(\phi_i) = \prod_{\sigma \in K_i(\mathbb{C})} \sigma(x_i)$$

$$\Rightarrow |\det(\phi_i)| = \prod_{\sigma \in K_i(\mathbb{C})} |\sigma(x_i)|$$

For $\tau \in K_i(\mathbb{C})$ we define $\overline{\tau} \in K_{\mathbb{C}}$ such that

$$\overline{\tau}(x_1, \dots, x_n) = \tau(x_i)$$

Then by Lemma (1.6)(ii) one has a bijection

$$\prod_{i=1}^{n} K_i(\mathbb{C}) \xrightarrow{\sim} K(\mathbb{C})$$

Therefore,

$$|\det(\phi_x)| = \prod_{i=1}^{n} |\det \phi_i| = \prod_{i=1}^{n} \prod_{\tau \in K_i(\mathbb{C})} \tau(x_i) = \prod_{i=1}^{n} \prod_{\tau \in K_i(\mathbb{C})} \overline{\tau}(x) = \prod_{\sigma \in K(\mathbb{C})} \sigma(x)$$

Hence, one only needs to show that

$$\#\left({}^{\mathcal{O}}/_{x\mathcal{O}}\right) = \prod_{\mathfrak{p} \in mSpec(\mathcal{O})} \#\left({}^{\mathcal{O}}/_{\mathfrak{p}}\right)^{ord_{\mathfrak{p}}(x)}$$

Now recall that $\mathcal{O}$ is a 1-dimensional Noetherian ring, let $\mathfrak{p} \in Spec\left({}^{\mathcal{O}}/_{x\mathcal{O}}\right)$ and suppose $x\mathcal{O} \subseteq \mathfrak{p}$ is not minimal, i.e

$$x\mathcal{O} \subseteq \mathfrak{p} \subsetneq \mathfrak{q} \quad \text{for some } \mathfrak{q} \in Spec\left({}^{\mathcal{O}}/_{x\mathcal{O}}\right)$$

Then

$$\pi^{-1}(\mathfrak{p}) \subsetneq \pi^{-1}(\mathfrak{q}) \text{ in } \mathcal{O}, \quad \text{where} \quad \pi : \mathcal{O} \twoheadrightarrow {}^{\mathcal{O}}\big/_{x\mathcal{O}}$$

Thus $ht(\pi^{-1}(\mathfrak{p})) = 1$ which contradicts the dimension of $\mathcal{O}$. Hence by Krull's principal ideal theorem,

$$ht(\mathfrak{p}) = 1 \quad \Rightarrow \quad dim\left({}^{\mathcal{O}}\big/_{x\mathcal{O}}\right) = 0$$

Thus ${}^{\mathcal{O}}\big/_{x\mathcal{O}}$ is an Artinian module. In particular, it has finite support.
Let $Supp\left({}^{\mathcal{O}}\big/_{x\mathcal{O}}\right) = \{\mathfrak{p}_1, \dots \mathfrak{p}_m\} \subseteq mSpec\left({}^{\mathcal{O}}\big/_{x\mathcal{O}}\right) (= Spec\left({}^{\mathcal{O}}\big/_{x\mathcal{O}}\right) < \infty)$. Again, by Lemma 1.6 in [1], one has

$${}^{\mathcal{O}}\big/_{x\mathcal{O}} \cong \bigoplus_{i=1}^{m} {}^{\mathcal{O}_{\mathfrak{p}_i}}\big/_{x\mathcal{O}_{\mathfrak{p}_i}}$$

In particular, there is a composite series

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = M$$

where

$${}^{M_i}\big/_{M_{i-1}} \quad \text{are simple Artinian } \mathcal{O}_{\mathfrak{p}_i}\text{-modules.}$$

Let $m \in {}^{M_i}\big/_{M_{i-1}}$, then by simplicity

$$0 \neq \mathcal{O}_{\mathfrak{p}_i} m \subseteq {}^{M_i}\big/_{M_{i-1}} \Rightarrow \mathcal{O}_{\mathfrak{p}_i} m = {}^{M_i}\big/_{M_{i-1}}$$

Now consider

$$Ann\left({}^{M_i}\big/_{M_{i-1}}\right) := I \hookrightarrow \mathcal{O}_{\mathfrak{p}_i} \twoheadrightarrow \mathcal{O}_{\mathfrak{p}_i} m = {}^{M_i}\big/_{M_{i-1}}$$
$$a \mapsto am$$

Then I is maximal by simplicity (if $I \subseteq \mathfrak{a} \subsetneq \mathcal{O}_{\mathfrak{p}_i}$ then ${}^{A}\big/_{I} \cong {}^{A}\big/_{\mathfrak{a}} \neq 0$ which is impossible) thus $I = \mathfrak{p}_i \mathcal{O}_{\mathfrak{p}_i}$ (since $\mathcal{O}_{\mathfrak{p}_i}$ is local) and one has

$${}^{M_i}\big/_{M_{i-1}} \cong {}^{\mathcal{O}_{\mathfrak{p}_i}}\big/_{\mathfrak{p}_i\mathcal{O}_{\mathfrak{p}_i}} \cong {}^{\mathcal{O}}\big/_{\mathfrak{p}_i}$$

Therefore,

$$\#\left({}^{\mathcal{O}}\big/_{\mathfrak{p}_i}\right)^m = \prod_{i=1}^{m} \#\left({}^{\mathcal{O}}\big/_{\mathfrak{p}_i}\right) = \prod_{i=1}^{m} \#\left({}^{M_i}\big/_{M_{i-1}}\right) = \prod_{i=1}^{m} \frac{\#(M_i)}{\#(M_{i-1})} = \frac{\#(M)}{\#(M_0)} = \#M$$

Since $ord_{\mathfrak{p}_i}(x) = length_{\mathcal{O}_\pi}\left({}^{\mathcal{O}_{\mathfrak{p}_i}}\big/_{\mathfrak{p}_i\mathcal{O}_{\mathfrak{p}_i}}\right) = m$, one finally gets

$$\#\left({}^{\mathcal{O}}\big/_{x\mathcal{O}}\right) = \prod_{i=1}^{m} \#\left({}^{\mathcal{O}}\big/_{\mathfrak{p}_i}\right) = \prod_{\mathfrak{p} \in mSpec(\mathcal{O})} \#\left({}^{\mathcal{O}}\big/_{\mathfrak{p}}\right)^{ord_{\mathfrak{p}}(x)}$$

□

1.2 Modules over reduced orders

In the following, we are only interested in reduced orders, we will investigate finitely generated $\mathcal{O}$-modules, over a given reduced order $\mathcal{O}$.

Definition 1.9. *Let $\mathcal{O}$ be a reduced order. A finitely generated $\mathcal{O}$-module H is said to have pure rank r if*

$$r = \dim_{\mathcal{O}_\mathfrak{q}} H_\mathfrak{q} \quad \text{for all minimal prime ideals } \mathfrak{q}$$

The non negative integer r is called the rank of H.

We have the following useful and characteristic proposition

Proposition 1.10. *Let $\mathcal{O}$ be a reduced order. Then an $\mathcal{O}$-module H is of pure rank r if and only if there exist elements $x_1, \dots, x_r$ of H such that:*

1. *$H/(\mathcal{O}x_1 + \dots + \mathcal{O}x_r)$ is a torsion $\mathbb{Z}$-module.*
2. *$\mathcal{O}x_1 + \dots + \mathcal{O}x_r$ is a free $\mathcal{O}$-module of rank r.*

We need the following short lemma:

Lemma 1.11. *Let $\mathcal{O}$ be a reduced order. For a finitely generated $\mathcal{O}$-module H we have the following:*

$$H \text{ is a torsion } \mathbb{Z}\text{-module} \Leftrightarrow H \text{ is of pure rank } 0.$$

Proof. Suppose H is a torsion $\mathbb{Z}$-module, i.e. $H \otimes_\mathbb{Z} \mathbb{Q} = 0$. Recall that since $\mathcal{O}$ is integral and flat over $\mathbb{Z}$, for all minimal prime $\mathfrak{q}$ of $\mathcal{O}$, $\mathfrak{q} \cap \mathbb{Z} = \{0\}$. Thus we can consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}_{(0)} = \mathbb{Q} \\ \big\downarrow & & \big\downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}_\mathfrak{q} \end{array}$$

Hence

$$\{0\} = (H \otimes_\mathbb{Z} \mathbb{Q}) \otimes_\mathbb{Q} \mathcal{O}_\mathfrak{q} = H \otimes_\mathbb{Z} \mathcal{O}_\mathfrak{q} \twoheadrightarrow H \otimes_\mathcal{O} \mathcal{O}_\mathfrak{q}$$
$$h \otimes x \mapsto h \otimes x$$

And we get

$$H \otimes_\mathcal{O} \mathcal{O}_\mathfrak{q} = H_\mathfrak{q} = \{0\}$$

Conversely, suppose that H has pure rank 0, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be minimal prime ideals of $\mathcal{O}$. Then by Lemma (1.6) $\mathfrak{q}_i = \mathcal{O} \cap \mathfrak{m}_i$ with $\mathfrak{m}_i \in$ m-Spec(K) and

$$K \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathfrak{q}_i}$$

One has that

$$H \otimes_{\mathcal{O}} K \cong \bigoplus_{i=1}^{n} H \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{q}_i} \cong \bigoplus_{i=1}^{n} H_{\mathfrak{q}_i} = \{0\}$$

On the other hand,

$$H \otimes_{\mathcal{O}} K = H \otimes_{\mathcal{O}} S^{-1}\mathcal{O} \cong S^{-1}H \cong H \otimes_{\mathbb{Z}} S^{-1}\mathbb{Z} = H \otimes_{\mathbb{Z}} \mathbb{Q}$$

□

Now let us prove Proposition 1.10.

Proof proposition 1.10. Suppose H is of pure rank $r > 0$, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be minimal prime ideals of $\mathcal{O}$, then as in the proof of Lemma 1.11 one has

$$K \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathfrak{q}_i} \quad \text{and} \quad H \otimes_{\mathcal{O}} K \cong \bigoplus_{i=1}^{n} H_{\mathfrak{q}_i}.$$

Let $\{\omega_{i_1}, \dots, \omega_{i_r}\}$ be a basis of $H_{\mathfrak{q}_i}$ (recall that $\dim_{\mathcal{O}_{\mathfrak{q}_i}} H_{\mathfrak{q}_i} = r$) and

$$\omega_j = (\omega_{1_j}, \dots, \omega_{n_j}) \in \bigoplus_{i=1}^{n} H_{\mathfrak{q}_i}.$$

Since $H \otimes_{\mathcal{O}} K = H \otimes_{\mathbb{Z}} \mathbb{Q}$, there exists a non-zero integer N and elements $x_1, \dots, x_r$ of H such that

$$N\omega_j = x_j \otimes 1 \quad \forall j = 1, \dots, r.$$

Hence

$$N \cdot \left({}^{H}\big/_{(\mathcal{O}x_1 + \cdots + \mathcal{O}x_r)} \right) = \{0\}$$

and thus ${}^{H}\big/_{(\mathcal{O}x_1 + \cdots + \mathcal{O}x_r)}$ is a torsion $\mathbb{Z}$-module. In particular, one has that

$$\operatorname{rank}(\mathcal{O}x_1 + \cdots + \mathcal{O}x_r) = r$$

By Lemma (1.11)

$${}^{H_{\mathfrak{q}_i}}\big/_{(\mathcal{O}x_1 + \cdots + \mathcal{O}x_r)_{\mathfrak{q}_i}} = \{0\} \quad \text{for all} \quad \mathfrak{q}_i$$

By abusive use of the exactness of the localization (which makes it commute with direct sums and images of morphisms)

$$\begin{aligned} (\mathcal{O}x_1 + \cdots + \mathcal{O}x_r)_{\mathfrak{q}_i} &= im(\bigoplus_{i=1}^{r} \mathcal{O}x_i \longrightarrow \mathcal{O})_{\mathfrak{q}_i} \\ &= im(\bigoplus_{i=1}^{r} (\mathcal{O}x_i)_{\mathfrak{q}_i} \longrightarrow \mathcal{O}_{\mathfrak{q}_i}) \end{aligned}$$

And again
$$(\mathcal{O}x_i)_{\mathfrak{q}_i} = im(\mathcal{O} \xrightarrow{\cdot x_i} \mathcal{O})_{\mathfrak{q}_i} \cong im(\mathcal{O}_{\mathfrak{q}_i} \xrightarrow{\cdot x_i} \mathcal{O}_{\mathfrak{q}_i}) = \mathcal{O}_{\mathfrak{q}_i} x_i$$
Therefore,
$$H_{\mathfrak{q}_i} = \mathcal{O}_{\mathfrak{q}_i} x_1 + \cdots + \mathcal{O}_{\mathfrak{q}_i} x_r$$
As H has pure rank r, $\{x_1, \ldots, x_r\}$ form a basis of $H_{\mathfrak{q}_i}$ and
$$\begin{aligned} a_1 x_1 + \cdots + a_r x_r = 0 &\Rightarrow a_1 = \cdots = a_r \text{ in } \mathcal{O}_{\mathfrak{q}_i} \quad \forall i \\ &\Rightarrow a_1 = \cdots = a_r \text{ in } \mathcal{O} \end{aligned}$$
Thus $\mathcal{O}x_1 + \cdots + \mathcal{O}x_r$ is a free $\mathcal{O}$-module of rank r.

Conversely, suppose that $H/(\mathcal{O}x_1 + \cdots + \mathcal{O}x_r)$ is a torsion $\mathbb{Z}$-module, then by Lemma (1.11) as done above,
$$H_{\mathfrak{q}_i} = \mathcal{O}_{\mathfrak{q}_i} x_1 + \cdots + \mathcal{O}_{\mathfrak{q}_i} x_r$$
Since $\{x_1, \ldots, x_r\}$ is a basis of $\mathcal{O}x_1 + \cdots + \mathcal{O}x_r$, $H_{\mathfrak{q}_i}$ is a free $\mathcal{O}_{\mathfrak{q}_i}$-module of rank r for all i, hence H is an $\mathcal{O}$-module of pure rank r. □

Finally, we prove the following two general lemmas that will be useful in the next talks.

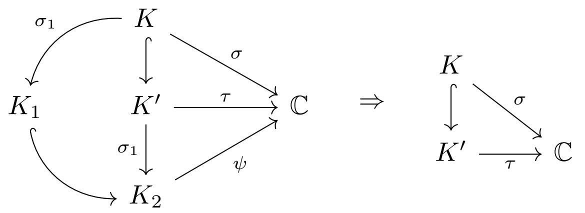
Lemma 1.12. *Let $\mathbb{Q} \subset K \subset K'$, where K, K' are finite dimensional reduced $\mathbb{Q}$-algebras.*
Recall that $K(\mathbb{C}) := \mathrm{Hom}_R(K, \mathbb{C})$, $K'(\mathbb{C}) := \mathrm{Hom}_R(K', \mathbb{C})$.
For $\sigma \in K(\mathbb{C})$ consider
$$K'(\mathbb{C})_\sigma = \{\tau \in K'(\mathbb{C}) \mid \tau \restriction_K = \sigma\}$$
Then for $x' \in K'$
$$\sigma\left(N_{K'/K}(x')\right) = \prod_{\tau \in K'(\mathbb{C})_\sigma} \tau(x')$$

Proof. Let σ be an element of $K(\mathbb{C})$. We proceed as follow:

case_1 : K' is a number field. (So is K)

Fix $\sigma_1 \in K'(\mathbb{C})_\sigma$ and set $K_1 = \sigma_1(K)$, $K_2 = \sigma_1(K')$. We define $K_2/K_1(\mathbb{C})$ to be the set of homomorphisms from K_2 to $\mathbb{C}$ over K_1, i.e.
$$K_2/K_1(\mathbb{C}) := \{\psi \in K_2(\mathbb{C}) \mid \psi \restriction_{K_1} \text{ is the canonical inclusion}\}.$$
Then one has a bijection
$$\begin{aligned} K_2/K_1(\mathbb{C}) &\cong K'(\mathbb{C})_\sigma \\ \psi &\mapsto \tau = \psi \circ \sigma_1 : \end{aligned}$$

Indeed, $\psi \circ \sigma_1 = \psi' \circ \sigma_1 \Rightarrow \psi = \psi'$ since σ_1 is an injective ring homomorphism, and surjectivity follows from the following: for an element y of K_2, there is an element $x' \in K'$ such that $y = \sigma_1(x')$ and so $\psi(y) = \psi \circ \sigma_1(x') \in \mathbb{C}$. Now observe that the two left and down right commutative blocks of the following diagram give



commutes.

Now let $x' \in K'$, we have that $\sigma_1(x') \in K_2$ and

$$N_{K_2/K_1}(\sigma_1(x')) = \prod_{\psi \in K_2/K_1(\mathbb{C})} \psi(\sigma_1(x')) = \prod_{\tau \in K'(\mathbb{C})_\sigma} \tau(x')$$

On the other hand,

$$\begin{aligned} N_{K'/K}(x') &= \det\left(\varphi_x : K' \to K'\right) \in K \\ \Rightarrow \quad \sigma(N_{K'/K}(x')) &= \det\left(\varphi_{\sigma_1(x)} : K_2 \to K_2\right) \\ &= N_{K_2/K_1}(\sigma_1(x')) \\ &= \prod_{\tau \in K'(\mathbb{C})_\sigma} \tau(x') \end{aligned}$$

case$_2$: K is an algebraic number field. (So K' is a finite product of number fields.)

Let $(x'_1, \ldots, x'_n) = x'$ and $x' \in K' \cong \bigoplus_{i=1}^n K'_i$. By multiplicativity of the norm,

$$N_{K'/K}(x') = \prod_{i=1}^n N_{K'_i/K}(x')$$

For all $\tau \in K'_i(\mathbb{C})_\sigma$ we define $\overline{\tau} \in K'(\mathbb{C})_\sigma$ such that $\overline{\tau}(x'_1, \ldots, x'_n) = \tau(x_i)$. Then by (**??**) in Lemma **??**, one has a bijection

$$\coprod_{i=1}^n K'_i(\mathbb{C})_\sigma \xrightarrow{\sim} K'(\mathbb{C})_\sigma$$
$$\tau \mapsto \overline{\tau}$$

Hence

$$\begin{aligned}\prod_{\overline{\tau}\in K'(\mathbb{C})_\sigma} \overline{\tau}(x') &= \prod_{i=1}^{n} \prod_{\tau\in K_i'(\mathbb{C})_\sigma} \tau(x') \\ &= \prod_{i=1}^{n} \sigma(N_{K_i'/K}(x')) \ \ \text{(by case 1)} \\ &= \sigma(\prod_{i=1}^{n} N_{K_i'/K}(x')) = \sigma(N_{K'/K}(x')).\end{aligned}$$

case$_3$: In general, K is a finite product of number fields.
let

$$K = \bigoplus_{i=1}^{n} K_i$$

and let $\mathfrak{p}_j = \{(x_1,\ldots,x_n) \in K \mid \ x_j = 0\}$ be the minimal prime ideal of K such that $K_{\mathfrak{p}_j} = K_j$. Let $K_j' := K_{\mathfrak{p}_j}'$, then one has

$$K' = \bigoplus_{j=1}^{n} K_j' \quad \text{and} \quad N_{K'/K}(x') = (N_{K_1'/K}(x_1'),\ldots,N_{K_n'/K}(x_n')) \in K$$

Since K is reduced, then again by (**??**) in Lemma (**??**) one has a bijection

$$\coprod_{i=1}^{n} K_j(\mathbb{C}) \xrightarrow{\sim} K(\mathbb{C})$$

$$\sigma_j \mapsto \sigma$$

Since $K_{\mathfrak{p}_j} = K_j$, for $x = (x_1,\ldots,x_n)$: $\sigma(x) = \sigma_j(x_j)$.
In particular,

$$\sigma(N_{K'/K}(x')) = \sigma((N_{K_1'/K}(x_1'),\ldots,N_{K_n'/K}(x_n'))) = \sigma_j(N_{K_j'/K}(x_j'))$$

Moreover,

$$\begin{array}{ccc} \bigoplus_{j=1}^{n} K_j & & \\ \downarrow & \searrow^{\sigma} & \\ \bigoplus_{j=1}^{n} K_j' & \xrightarrow[\tau]{} & \mathbb{C} \end{array} \quad \text{commutes} \quad \Leftrightarrow \quad \begin{array}{ccc} K_j & & \\ \downarrow & \searrow^{\sigma_j} & \\ K_j' & \xrightarrow[\tau_j]{} & \mathbb{C} \end{array} \quad \text{commutes}$$

Thus

$$\begin{aligned}\prod_{\tau\in K'(\mathbb{C})_\sigma} \tau(x') &= \prod_{\tau_j\in K_j'(\mathbb{C})_{\sigma_j}} \tau_j(x_j') \\ &= \sigma_j(N_{K_j'/K}(x_j')) \ \ \text{(by case 2)} \\ &= \sigma(N_{K'/K}(x'))\end{aligned}$$

□

Lemma 1.13. *Let $p_1, \dots, p_n$ be distinct prime numbers and let $m_1, \dots, m_n$ be positive integers. Then there is an algebraic number field K such that*

$$\left[{}^{\mathcal{O}_K}\big/_{\mathfrak{p}} : {}^{\mathbb{Z}}\big/_{(p_i)}\right] \geq m_i$$

for $i = 1, \dots, n$ and all $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K)$ with $\mathfrak{p} \cap \mathbb{Z} = p_i\mathbb{Z}$.

Proof. Let $f_i \in {}^{\mathbb{Z}}\big/_{(p_i)}[X]$ be an irreducible monic polynomial of degree m_i. Indeed, such polynomial exists: since the multiplicative group of non-zero elements of any finite field is cyclic; if we take $K' = \mathbb{F}_{p_i^{m_i}}$ and β be a generator of the multiplicative group of K, by the primitive element theorem, we have that $K = \mathbb{F}_{p_i}(\beta)$. In particular, the minimal polynomial of β over $\mathbb{F}_{p_i}$, which is irreducible, must have the same degree as $[\mathbb{F}_{p_i}(\beta) : \mathbb{F}_{p_i}] = [K' : \mathbb{F}] = m_i$.
Now let $F_i \in \mathbb{Z}[X]$ such that $\overline{F_i} = f_i$, and

$$F(X) = \prod_{i=1}^{n} F_i(X)$$

Let K be the splitting field of F. Then if α_i is a root of $F_i(X)$, $\alpha_i \in \mathcal{O}_K$ since α_i is a root of $F \in \mathbb{Z}[X]$. Let $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K)$ lying over $p_i\mathbb{Z}$, and let $\overline{\alpha_i}$ be the class of α_i in ${}^{\mathcal{O}_K}\big/_{\mathfrak{p}}$. Then ${}^{\mathbb{Z}}\big/_{(p_i)} \subset {}^{\mathcal{O}_K}\big/_{\mathfrak{p}}$ hence, as $f_i(\overline{\alpha_i}) = 0$ and f_i irreducible, it is the minimal polynomial of $\overline{\alpha_i}$ and therefore

$$\left[{}^{\mathcal{O}_K}\big/_{\mathfrak{p}} : {}^{\mathbb{Z}}\big/_{(p_i)}\right] \geq \left[{}^{\mathbb{Z}}\big/_{(p_i)}(\overline{\alpha_i}) : {}^{\mathbb{Z}}\big/_{(p_i)}\right] = m_i$$

□

2 Chow groups

2.1 Geometric Chow groups

Definition 2.1 (Weil divisors)**.** *The group of Weil divisors (or algebraic cycles of co-dimension 1) denoted $Z^1(\mathcal{O})$ is the free abelian group with basis consisting of all maximal prime ideals of $\mathcal{O}$ i.e*

$$Z^1(\mathcal{O}) = \bigoplus_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} \mathbb{Z}[\mathfrak{m}]$$

Elements of $Z^1(\mathcal{O})$, also called 1-codimensional cycles are simply formal sums

$$\sum_{\mathfrak{m}} n_{\mathfrak{m}} \cdot [\mathfrak{m}]$$

where $n_{\mathfrak{m}} \in \mathbb{Z}$ and $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$.

Now recall that from the last section of talk 1 (proposition (4.3) or [1] p.10) that for a maximal ideal $\mathfrak{m} \in$ m-Spec($\mathcal{O}$) the order map (defined on $\mathcal{O} \setminus \{0\}$) extends to a unique group homomorphism

$$ord_{\mathfrak{m}} : K^{\times} \longrightarrow \mathbb{Z}$$

Such that for every non-zero $x = \frac{y}{z} \in K$,

$$ord_{\mathfrak{m}}(x) = \text{length}_{\mathcal{O}_{\mathfrak{m}}} \left({\mathcal{O}_{\mathfrak{m}}}\big/{x\mathcal{O}_{\mathfrak{m}}} \right) = ord_{\mathfrak{m}}(y) - ord_{\mathfrak{m}}(z)$$

Definition 2.2 (divisors)**.** *For any non-zero element of K we define its divisor to be*

$$div(x) = \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} ord_{\mathfrak{m}}(x)[\mathfrak{m}]$$

The order of a regular element is indeed an element of $Z^1(\mathcal{O})$ and the order homomorphism extends uniquely into a group homomorphism

$$div : K^{\times} \longrightarrow Z^1(\mathcal{O})$$

(by inherited properties of the order map).

Definition 2.3 (Rational equivalence)**.** *Two cycles α, β of codim 1 in $Z^1(\mathcal{O})$ are said to be rationally equivalent if their difference is a divisor of some non-zero regular element in K, i,e*

$$\alpha \sim \beta \Leftrightarrow \alpha - \beta \in div(K^{\times})$$

We denote $B^1(\mathcal{O}) := div(K^{\times}) \leq Z^1(\mathcal{O})$ the subgroup consiting of cycles that are rationally equivalent to 0. Then one can finally define chow groups as the following:

Definition 2.4 (Chow groups of codim 1)**.** *We define the Chow group of codimension 1 of a reduced order $\mathcal{O}$ to be the group*

$$CH^1(\mathcal{O}) = {Z^1(\mathcal{O})}\big/{B^1(\mathcal{O})}$$

To give some ground floor and motivations behind the definitions of Chow groups, assume that $\mathcal{O}$ is normal and let K be its quotient field. (For example, take $\mathcal{O}_{\mathfrak{m}}$ to be the maximal order in some number field.) then $\mathcal{O}_{\mathfrak{m}}$ is a DVR (in fact, this stands for every height 1 prime ideal) thus any ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathfrak{m}}$ is a power of the maximal ideal $\mathfrak{m}\mathcal{O}_{\mathfrak{m}}$, that is $v_{\mathfrak{m}}(\mathfrak{a})$ where $v_{\mathfrak{m}}$ is the valuation on $\mathcal{O}$. Moreover, $v_{\mathfrak{m}}(\mathfrak{a}) = \text{length}_{\mathfrak{m}}(\mathcal{O}/\mathfrak{a})_{\mathfrak{m}}$ induces a bijection from the group of fractional ideals of $\mathcal{O}$ (non-zero finitely-generated $\mathcal{O}$-submodule) to the group of cycles of codimension 1:

$$\varphi : D(\mathcal{O}) \xrightarrow{\sim} Z^1(\mathcal{O})$$
$$\mathfrak{a} \longmapsto \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} v_{\mathfrak{m}}(\mathfrak{a})[\mathfrak{m}]$$

Indeed, since $\mathcal{O}$ is a Dedekind domain, this follows from

Proposition 2.5 ([4], Ch. I, Cor. 3.9)**.** *Let $\mathcal{O}$ be a Dedekind domain. Every fractional ideal $\mathfrak{a}$ admits a unique representation as a product*

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$$

with $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$ and $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ for almost all $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. In other words, $D(\mathcal{O})$ is the free Abelian group on the set of nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

One can easily see that under this bijection, the image of the subgroup of principal ideals of $\mathcal{O}$ is precisely $B^1(\mathcal{O})$. Indeed, for a non-zero $x = \frac{y}{z} \in K$

$$\begin{aligned}\varphi(x\mathcal{O}) &= \varphi\left(\frac{y}{z}\mathcal{O}\right) = \varphi(y\mathcal{O}) - \varphi(z\mathcal{O}) \\ &= \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} v_{\mathfrak{m}}((y))[\mathfrak{m}] - \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} v_{\mathfrak{m}}((z))[\mathfrak{m}] \\ &= \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} ord_{\mathfrak{m}}(y)[\mathfrak{m}] - \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} ord_{\mathfrak{m}}(z)[\mathfrak{m}] \in B^1(\mathcal{O}).\end{aligned}$$

Thus one has a group isomorphism

$$Cl(\mathcal{O}) \cong CH^1(\mathcal{O}).$$

Where $Cl(\mathcal{O})$ denotes the ideal class group of $\mathcal{O}$.

Remark 2.6. *Any non-zero ring has a non-trivial Chow group (since any non-zero ring has at least one maximal ideal).*

2.2 Arithmetic Chow groups

In a similar fashion, we define Arithmetic Chow groups as a generalization of geometric Chow groups, where an "analytic" data is added.

Definition 2.7 (Arithmetic divisors)**.** *The group of arithmetic divisor (or arithmetic cycles of co-dimension* 1*) denoted $\widehat{Z}^1(\mathcal{O})$ is the direct sum of the group of Weil divisors and an $\mathbb{R}$-vector space generated by $K(\mathbb{C})$. i.e*

$$\widehat{Z}^1(\mathcal{O}) = Z^1(\mathcal{O}) \times \left(\bigoplus_{\sigma \in K(\mathbb{C})} \mathbb{R}[\sigma] \right)$$

Elements of $\widehat{Z}^1(\mathcal{O})$, also called 1*-codimensional arithmetic cycles are pairs of the form*

$$\left(\sum_{\mathfrak{m}} n_{\mathfrak{m}} \cdot [\mathfrak{m}], \sum_{\sigma} \lambda_{\sigma}[\sigma] \right)$$

where $n_{\mathfrak{m}} \in \mathbb{Z}$, $\lambda_{\sigma} \in \mathbb{R}$ and $\mathfrak{m} \in \text{m-Spec}(\mathcal{O})$.

Definition 2.8 (Arithmetic divisors)**.** *For any non-zero element of K we define its arithmetic divisor to be*

$$\widehat{div}(x) = \left(\sum_{\mathfrak{m}} ord_{\mathfrak{m}}(x)[\mathfrak{m}], \sum_{\sigma} -\log|\sigma(x)|^2[\sigma] \right) \in \widehat{Z}^1(\mathcal{O})$$

By a similar argument, one sees that the induced map

$$\widehat{div} : K^{\times} \longrightarrow \widehat{Z}^1(\mathcal{O})$$

is a homomorphism of Abelian groups.

Definition 2.9 (Arithmetic Chow groups of codim 1)**.** *We define the arithmetic Chow group of codimension* 1 *of a reduced order $\mathcal{O}$ to be the group*

$$\widehat{CH}^1(\mathcal{O}) = {}^{\widehat{Z}^1(\mathcal{O})}\big/_{im(\widehat{div})}$$

Now, for an element $(D, g) \in \widehat{Z}^1(\mathcal{O})$ we set

$$\widehat{deg}(D, g) = \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} n_{\mathfrak{m}} \log(\# \left({}^{\mathcal{O}}/_{\mathfrak{m}} \right)) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} g_{\sigma} \in \mathbb{R}$$

Then for $x \in K^{\times}$

$$\begin{aligned} \widehat{deg} \circ \widehat{div}(x) &= \sum_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} ord_{\mathfrak{m}}(x) \log(\# \left({}^{\mathcal{O}}/_{\mathfrak{m}} \right)) - \sum_{\sigma \in K(\mathbb{C})} \log|\sigma(x)| \\ &= -\log \left(\prod_{\mathfrak{m} \in \text{m-Spec}(\mathcal{O})} (\# \left({}^{\mathcal{O}}/_{\mathfrak{m}} \right))^{-ord_{\mathfrak{m}}(x)} \cdot \prod_{\sigma \in K(\mathbb{C})} |\sigma(x)| \right) \\ &= 0 \qquad \text{(By the product formula in Theorem ??)} \end{aligned}$$

Thus the map factors through $\widehat{CH}^1(\mathcal{O})$ and we have

$$\begin{array}{ccc} \widehat{Z}^1(\mathcal{O}) & \xrightarrow{\widehat{deg}} & \mathbb{R} \\ \downarrow & \nearrow_{\widehat{deg}'} & \\ \widehat{CH}^1(\mathcal{O}) & & \end{array} \qquad \text{commutes.}$$

Exercise 2.10. *Suppose $\mathcal{O} = \mathbb{Z}$, $K = \mathbb{Q}$. Is the $\widehat{deg}'$ map an isomorphism ?*

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